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Comment on "Hamilton's Principle, Hamilton's Law—6^n Correct Formulations"

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THE authors' main point[1] would seem to be that there are six correct formulations of Hamilton's law, based on the six combinations of vanishing end point variations shown in their Fig. 1, and there is a strong implication throughout the paper that there are only six correct formulations. However, there are exactly ten additional combinations involving the vanishing of the quantities δq and $\delta\dot{q}$ at the end points; and it will be shown that solutions exist using these other combinations.

Begin with the principle of virtual work for the one-degree-of-freedom system discussed in the paper:

$$\int_0^1 \left[f - kq - \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q})\right]\delta q \mathrm{d}t = 0 \tag{1}$$

The real displacement, q, must ultimately satisfy the initial conditions, but there are no requirements on the virtual displacements or velocities at either end point.

One integration by parts gives

$$\int_0^1 (f - kq)\,\delta q \mathrm{d}t + \int_0^1 m\dot{q}\delta\dot{q}\mathrm{d}t - m\dot{q}\delta q\Big|_0^1 = 0 \tag{2}$$

which is the authors' Eq. (2), showing the equivalence between the principle of virtual work and Hamilton's law of varying action. Equations (1) and (2) are completely equivalent assuming sufficient continuity of the integrands; the choice depends simply on which equation gives the most simple calculations. If the integrands are not continuous, which might be the case in a time domain finite element formulation, then Eq. (1) should include additional terms to account for discontinuities.

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Now consider the authors' first example described by their Eq. (10), with an admissible trial function given by Eq. (14). The virtual displacements are given in Eqs. (15), (17), (22), (24), (26), and (28). If these virtual displacements, along with Eq. (14), are substituted into Eq. (1) or Eq. (2) above, the results for a_2 agree with answers presented in the paper. This would seem to suggest that there are not 16 different formulations of Hamilton's law. There is only one formulation—the authors' Eq. (2), which is a derivative of the principle of virtual work—with 16 different types of virtual displacements when classified according to vanishing of certain end point quantities. The question is not which one of the many formulations to use. The only question is what is the best choice from among many virtual displacements for use in the one principle of virtual work.

Consider some of the other possibilities for virtual displacements with the same trial function and the principle of virtual work. If $\delta q_1 = b(t^4 - 2t^3 + t^2)$, for which all four end point values of δq and $\delta\dot{q}$ vanish, then $a_2 = 1.5$. If $\delta q_2 = b(\frac{1}{2}t^2 - \frac{1}{3}t^3)$, for which three end point quantities vanish, then $a_2 = 2.1$. If $\delta q_3 = b(t)$, for which only one end point δq vanishes, then $a_2 = 2$. Finally, if $\delta q_4 = b[1-(8/7)t]$, for which no end point values vanish, then $a_2 = 5/6$.

The authors compare their six solutions through a coefficient defined in their Eq. (30). It is easier to simply define a least-square error given by

$$E = \int_0^1 (q_{\text{exact}} - \bar{q}_{\text{approx}})^2 \mathrm{d}t = \frac{1}{7} - \frac{1}{3}a_2 + \frac{1}{5}a_2^2$$

with the best answer giving the lowest value of E. Such a comparison is shown in Table 1.

The four additional cases are all better than the worst of the authors' results (F1); and δq_4, which says nothing about any of the end point quantities, provides a better result than the best of the author's formulations (F4). The virtual displacement δq_4 has been contrived to provide the lowest value for E, but the point is that the principle of virtual work permits such a δq while none of the claimed exclusive six correct formulations would have admitted any of the four additional cases, including the one that gives the best result.

The authors claim that their variations follow the physical path of the system. However, using the procedure clearly described in the paper, this is strictly true only for their F1 formulation. For all other cases, they take the variation of an artificial trial function, such as their Eq. (17). The artificial trial function is then discarded, and the real displacement substituted into Hamilton's law is always Eq. (14). There is an alternative procedure which might have been used. Select a real displacement and associated variation, such as Eq. (17), determine the consequences of satisfying Hamilton's law, and then adjust coefficients as necessary to satisfy initial conditions. With this approach, one might say that the virtual displacement is the variation of the real displacement, and this is the logic which must be used to precisely justify the use of Hamilton's Principle with the F2 formulation which satisfies admissibility requirements for the principle. However, the reasoning is convoluted, unnecessary, and has obscured the really important fact that what matters is the virtual displacement. With the risk of redundancy, consider one final example to illustrate the point. Begin with the real displacement $q = a_0 + q_0(t - \frac{1}{2}t^2) + \frac{1}{2}q_1 t^2$ and virtual displacement $\delta q = b(t^2)$. This is the authors' F3 real displacement and F1 virtual displacement, and it would seem

Table 1

Virtual displacement	F1	F4	δq_1	δq_2	δq_3	δq_4
Value of a_2	2.25	0.75	1.5	2.1	2.0	0.833
Value of E	0.4054	0.00536	0.0929	0.3249	0.2762	0.00397

that never the twain could meet according to the concepts in the paper. However, substitution into the principle of virtual work, followed by satisfaction of initial conditions, gives the F1 result in the paper. One thing should now be clear. For a given trial function, the answer depends upon the choice of virtual displacement; and the authors' Eq. (13), expanded if necessary, should be written directly for δq, with coefficients adjusted to give any desired properties to δq.

Finally, it should be reiterated that the virtual and real displacements need not be in the same space of functions. This point was possibly first made in Ref. 2 in 1975 and then demonstrated in Ref. 3. Insisting on the same space sometimes clouds the issue and often makes it impossible to achieve best results, as seen by the fact that the authors' F1 result is their worst. The authors recognize this fact, as evidenced by the last paragraph of their discussion, in which they state "...Eq. (39) shows a further possible separation between the actual approximate displacements and their variations and more possible formulations." However, it must be noted that whether q and δq are in the same or different spaces has nothing whatever to do with the existence of exactly 16 different types of virtual displacements. The classification is based on vanishing of certain virtual quantities at the end points; it is not based on the relationship between q and δq. Therefore, it is misleading to suggest that the use of different spaces will lead to more possible formulations.

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Reply by Authors to C. V. Smith Jr.

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We would like to thank Professor Smith for his interest in our paper[1] and for his interesting remarks.

The main point of the Comment is that there are more than six correct formulations of Hamilton's Law. At this point, we do not see any contradiction between the paper[1] and the Comment. However, we cannot agree with the statement that "there is a strong implication throughout the paper that there are only six correct formulations." In the abstract of the paper one can read: "The paper centers around 6^n correct formulations of Hamilton's Law. Actually, as will be shown, there are more. The principle of virtual work, which is equivalent to Hamilton's Law, has many possibilities because the coordinates and their variations can be regarded as being mutally independent."[1]

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Professor Smith has shown some of these "many possibilities" involving the vanishing of the state variables at the end points. Clearly, he has obtained them by exploiting the mutual independence between the coordinates and their variations. However, one can obtain, for example, an infinite number of possibilities, by requiring that the variations vanish in some points of the time region—not just combinations formed from the end point variations.

From both Ref. 1 and the Comment it follows that in the application of Hamilton's Law the variations can be chosen quite freely. This is not the case with Hamilton's Principle. This case is obtained from the law "if the final and initial coordinates and the time be given."[2,3] Hence, the variations of the end displacements must vanish. As can be seen, this requirement was given by Hamilton in a verbal form. It seems to the authors that this is the reason why researchers have attempted to use the incomplete and hence incorrect formulation of Hamilton's Principle for direct calculations. To show a possible result of such application, a very simple example follows: Calculate by direct application of Hamilton's Principle the free falling of a mass point.

The incomplete Hamilton's Principle usually given in the literature is

$$\delta I = \delta \int_0^1 L\,dt = 0; \qquad L = T - V \tag{1}$$

For the given case one obtains

$$T = \tfrac{1}{2} m\dot{q}^2; \qquad V = -mgq \tag{2}$$

An admissible trial function for the displacement, built up from the polynomial

$$\bar{q} = a_0 + a_1 t + a_2 t^2 \tag{3}$$

and which fulfills the initial conditions $q_0 = \dot{q}_0 = 0$, is

$$\bar{q} = a_2 t^2 \tag{4}$$

Substitution of Eqs. (2) and (4) into Eq. (1) yields,

$$\bar{I} = \int_0^1 \left(\frac{1}{2} m\dot{q}^2 + mgq\right) dt = \frac{1}{3}(2ma_2^2 + mga_2) \tag{5}$$

$$\frac{\partial \bar{I}}{\partial a_2} = 0; \quad \rightarrow a_2 = -\frac{g}{4}; \quad \rightarrow \bar{q} = -\frac{g}{4} t^2 \tag{6}$$

Application of the incorrect formulation of Hamilton's Principle gave a monstrously incorrect result.

Now, the expression given in the paper,[1] built up from the same trial function of Eq. (3), for the correct formulation F2 is

$$\bar{q} = \bar{q}_0 (1 - t) + \bar{q}_1 t + a_2 (t^2 - t) \tag{7}$$

We pretend that $\bar{q}_0$ and $\bar{q}_1$ are known. By substitution of Eq. (7) into Eq. (1), calculation of Hamilton's integral and equating to zero the first derivative of this integral with respect to a_2 only, one obtains

$$a_2 = g/2 \tag{8}$$

Substitution of Eq. (8) into Eq. (7) and fulfillment of the actual initial conditions yields

$$q = (g/2) t^2 \tag{9}$$

which is obviously the correct result.

In Ref. 4 the correct formulation of Hamilton's Principle was used to develop time-finite elements. Bailey[5] and others